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1 Series Basics

Definition 1.1. A *series* is an "infinite sum" $\sum_{n=1}^{\infty} a_n$ for some *terms* a_n. We may begin the summation at any index, but $n = 0$ or $n = 1$ are preferred. When the initial index is obvious from context (or if it doesn't matter), I may be lazy and just write $\sum a_n$ instead. The *Nth partial sum* of a series $\sum_{n=1}^{\infty} a_n$ (or $\sum_{n=0}^{\infty} a_n$) is the finite sum $S_N := \sum_{n=1}^{N} a_n$ (or $S_N := \sum_{n=0}^{N} a_n$).

Definition 1.2. We say that a series $\sum a_n$ *converges* if its sequence of partial sums $\{S_N\}$ converges, and then we say that the sum of the sequence is the limit of the S_N. Otherwise, the series *diverges*. If $\lim_{N\to\infty} S_N$ is $\pm\infty$, then we may say that the series *diverges to* $\pm\infty$. We often write $\sum a_n < \infty$ if the series converges, and write $\sum a_n = \pm\infty$ if the series diverges to $\pm\infty$. If the series converges to S, then the Nth (absolute) error is defined as $E_N = |S - S_N|$.

Note that the (convergence/divergence) behavior of a series only depends on the "tail". That is, if $\sum_{n=1}^{N} a_n$ converges, then so do $\sum_{n=100}^{N} a_n$, $\sum_{n=1000000}^{N} a_n$, etc., and vice versa.

Example 1.3. Here are some examples of various series:

1. A *geometric* series is a series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$. a is the *initial term* and r is the *common ratio*. This series converges exactly when $|r| < 1$, in which case it has sum $\frac{a}{1-r}$. For example, the series $3 - 3\cdot(1/2)^2 + 3\cdot(1/2)^4 - 3\cdot(1/2)^6 + \ldots$ is geometric with $a = 3$ and $r = -((1/2)^2)$, so it converges and has sum $\frac{3}{1+(1/4)} = 12/5$.

2. The *harmonic series* is the series $\sum_{n=1}^{\infty} 1/n$, and it diverges to ∞.

3. An *alternating* series has terms that alternate sign, such as the *alternating harmonic series* $1 - 1/2 + 1/3 - 1/4 + \ldots$. Unlike the harmonic series, this series converges.

4. A p-series is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This converges when $p > 1$, and diverges otherwise. Note that the $p = 1$ case is the harmonic series.

5. We can generalize the above to series of the form $\sum_{n=3}^{\infty} \frac{1}{n^a (\log n)^b}$. This converges when $a > 1$ or when $a = 1$ and $b > 1$, and diverges otherwise.

6. We can consider any finite sum as a series by just adding infinitely many 0 terms. For instance, $1 + 2 + 3 + 4$ is a series $1 + 2 + 3 + 4 + 0 + 0 + \ldots$.

7. The series $\sum_{n=1}^{\infty} n$ obviously diverges to ∞ (the partial sums keep getting larger and larger).

8. Note that it is possible for a series to diverge, but not to $\pm\infty$. Consider the series $\sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+\ldots$; the partial sums alternate between 1 and 0, so neither converge nor diverge to ∞.

Notice that because the convergence of a series depends on its partial sums, there is *a priori* no reason that the sum of a series should stay the same if its terms are rearranged (because the sequence of partial sums will change). Consider the following example:

Example 1.4. Recall that the alternating harmonic series $1-1/2+1/3-1/4+\ldots$ converges; it in fact converges to $\log(2)$. Suppose that we rearrange the series as follows:

$$\left(\frac{1}{1}-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\left(\frac{1}{7}-\frac{1}{14}-\frac{1}{16}\right)+\ldots.$$

If we ignore the parentheses, we see that the terms of this new series are indeed the terms of the original series, but in a different order. But looking at the first two terms in each triplet indicated by the parentheses, the series equals

$$\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{10}-\frac{1}{12}\right)+\left(\frac{1}{14}-\frac{1}{16}\right)+\ldots=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)=\frac{1}{2}\log(2).$$

So we've added the terms in a different order, but got half of the original sum.

This example shows why it's important to make the following definition:

Definition 1.5. A series $\sum a_n$ is *absolutely convergent* if $\sum|a_n|$ converges. If $\sum a_n$ converges but $\sum|a_n|$ diverges, then the series is only *conditionally convergent*.

Of course, if a series with nonnegative terms converges, then it is absolutely convergent.
Absolutely convergent series are important for two reasons:

Theorem 1.6. If a series is absolutely convergent, then it is convergent.

This is useful because many of the convergence tests we will see later only work if all terms are nonnegative. Of course, a series of the form $\sum|a_n|$ has only nonnegative terms, so such tests will apply to this series even if they don't apply to the original series $\sum a_n$.

Theorem 1.7 (Riemann Rearrangement Theorem)**.** If a series $\sum a_n$ is absolutely convergent and has sum S, then any rearrangement of the series also converges and has sum S. Conversely, if the series is only conditionally convergent, then this is not true.

Therefore it is useful to deal with absolutely convergent series, so that we may rearrange their terms at will. Luckily, most of the convergence tests we will use also apply to absolute convergence.

We can also add and scale series: if $\sum a_n$ and $\sum b_n$ are convergent series, and c is any real number, then $\sum(a_n+b_n)$ and $\sum ca_n$ are convergent and equal $\sum a_n+\sum b_n$ and $c\sum a_n$, respectively. Conversely, if $\sum a_n$ diverges, then so does $\sum ca_n$ for any scalar $c\neq 0$ (why is the corresponding statement for sums of two series not true?).

2 Convergence Tests

It is intuitively obvious that if a series $\sum a_n$ converges, then the terms a_n must converge to 0—the idea is that the series cannot converge if the a_n are "large", since then the partial sums will be going to ∞ or jumping around. Therefore:

Theorem 2.1. If the terms a_n do not converge to 0, then the series $\sum a_n$ diverges.

Note that the converse is not true: consider the harmonic series.

As with improper integrals, we have direct comparison tests:

Theorem 2.2. Suppose two series $\sum a_n$ and $\sum b_n$ have nonnegative terms, and $a_n \leq b_n$ for all n. Then if $\sum b_n$ converges, $\sum a_n$ converges as well. Also, if $\sum a_n$ diverges, then $\sum b_n$ diverges as well.

We will usually compare to a geometric series or p-series. For instance, $\sum \frac{2\sin^2(n)}{n^2+1}$ converges upon being compared to $\sum \frac{2}{n^2} < \infty$.

Even better is the *limit comparison test*:

Theorem 2.3. Suppose two series $\sum a_n$ and $\sum b_n$ have positive terms, and suppose $\lim_{n\to\infty} \frac{a_n}{b_n}$ converges to some positive constant c (in particular, $c \neq 0, \infty$). Then the series $\sum a_n$ and $\sum b_n$ have the same behavior: they both converge or both diverge.

This theorem essentially generalizes the "hierarchy of functions" heuristic that we've seen before—it allows us to look at the "dominant" part of the terms that occur in the series, and ignore all other terms. Here is a typical application:

Example 2.4. The series $\sum_{n=1}^{\infty} \frac{n^2+4n+(-1)^n}{(1/2)n^4+100000\log(x)-e^{-x}-27}$ converges via a limit comparison test with the convergent p-series $\sum_{n=1} \frac{1}{n^2}$. Indeed, the limit $\lim_{n\to\infty} \frac{n^4+4n^3+(-1)^n n^2}{(1/2)n^4+100000\log(x)-e^{-x}-27}$ is $\frac{1}{2}$, since the dominant term in the numerator is n^4, and the dominant term in the denominator is $(1/2)n^4$ (the other terms are lower-degree polynomials, logarithms, or bounded).

The *ratio* and *root* tests are also particularly important, because they do not require the terms of the series to be nonnegative.

Theorem 2.5. Suppose we are given a series $\sum a_n$. Then if the limit $\lim_{n\to\infty}|a_{n+1}/a_n|$ exists and equals C: if $C < -1$, then the series converges; if $C = 1$, the test is inconclusive; if $C > 1$, the series diverges. The same statement is true with the preceding limit replaced by the limit $\lim_{n\to\infty}|a_n|^{1/n}$.

Remark 2.6. Note that the ratio and root tests actually test for *absolute* convergence (make sure you see why!).

Remark 2.7. The root test is stronger than the ratio test, in the sense that any series that we can analyze (i.e. determine convergence/divergence/inconclusive) using the ratio test, we can also analyze using the root test to get the same result. On the other hand, the ratio test is almost always easier to apply, so try to start with that one (unless it's really obvious that you can use the root test, such as when the terms a_n already are nth powers).

Example 2.8. The series $\sum \frac{n!}{n^n}$ converges using the ratio test: we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

As $n \to \infty$, the denominator of the last fraction goes to e, so $\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow{n\to\infty} e^{-1} < 1$.

It is also possible to do this with the root test, but it is a lot harder.

Example 2.9. Note that "inconclusive" in the ratio (or root) test really means "inconclusive": if the limit C in the test is 1, the series could exhibit any type of behavior. For instance, $\sum \frac{1}{n^2}$ has $C = 1$ and converges absolutely; $\sum \frac{(-1)^n}{n}$ has $C = 1$ and converges conditionally; $\sum \frac{1}{n}$ has $C = 1$ and diverges.

Finally, we discuss the integral and alternating series tests, which are useful in limited situations.

Theorem 2.10. Suppose $f : [1, \infty) \to \mathbf{R}$ is continuous, positive and decreasing, and let $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ is convergent exactly when $\int_1^{\infty} f(x)dx$ is convergent. Also, the Nth error E_N satisfies

$$\int_{N+1}^{\infty} f(x)dx \le E_N \le \int_N^{\infty} f(x)dx,$$

so the error is bounded by the tail of the integral.

For example, one can show that the p-series converges for $p > 1$ and diverges for $0 < p < 1$ using the integral test.

Theorem 2.11. If the series $\sum a_n$ is alternating, then it converges if $|a_n|$ decreases monotonically to 0 (the latter condition meaning that $\lim_{n\to\infty}|a_n| = 0$). Moreover, the Nth error E_N satisfies $E_N \le |b_{N+1}|$.

The application of the alternating series test is straightforward, but there are a few caveats. First, keep in mind that the alternating theorem says *nothing* about divergence: we do not give any conditions for an alternating series to be divergent. Second, note that one can often disguise a series to be alternating, or vice-versa. For instance, the series $\sum_{n=0}^{\infty} \cos(n\pi)e^{-n}$ is alternating, since $\cos(n\pi)$ is -1 if n is odd, and 1 if n is even. Conversely, series like $\sum_{n=0}^{\infty} \frac{(-1)^n \cos(n\pi)}{e^n}$ are not alternating, even though they might appear to be.

3 Power Series

Definition 3.1. A *power series* is a series of the form $\sum c_n(x-a)^n$, where the c_n are the *coefficients* of the power series (as opposed to the *terms*, which are the $c_n(x-a)^n$). The *center* of the series is a.

For a power series $\sum c_n(x-a)^n$, we want to consider its convergence/divergence behavior as x varies (but the c_n and a are fixed). It is clear that when $x=a$, the power series converges and equals 0. For other x, the series may or may not converge. But we have the following theorem:

Theorem 3.2. Every power series $\sum c_n(x-a)^n$ has a well-defined *radius of convergence* R: that is, there is a unique $R \geq 0$ (possibly $R=\infty$) such that if $|x-a|<R$, then it converges *absolutely*, and if $|x-a|>R$, then it diverges. In particular, the radius of convergence does not tell us what happens when $x=a-R$ or $x=a+R$.

In the special cases $R=0$ and $R=\infty$, we mean that the power series converges only if $x=a$ and that it converges for all real x, respectively.

Definition 3.3. The *interval of convergence* is the set of all x such that the power series $\sum c_n(x-a)^n$ converges. By the above theorem, if the radius of convergence is $R \neq 0$ or ∞, then the interval of convergence has the form $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$.

Notice that if x is strictly within the radius of convergence of a power series (i.e. not a boundary point), then the series converges absolutely at x. This may no longer be true if x is a boundary point.

Example 3.4. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n4^n}$. We have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|n}{4(n+1)}$, which converges to $\frac{|x|}{4}$. The ratio test tells us that the power series converges absolutely if $|x|<4$, and diverges if $|x|>4$. Therefore $R=4$, and we need to check the endpoints ± 4. At $x=4$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges. At $x=-4$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Therefore the interval of convergence is $(-4,4]$. Also, notice that the series does not converge absolutely at $x=4$.

The technique shown in the above example is typical: we take the relevant limit in the ratio or root test, and determine for which x that limit is strictly less than 1 or strictly greater than 1. For x such that the limit equals exactly 1, we know those will be the endpoints of our radius of convergence; those need to be checked by hand using a different test, such as (limit) comparison, alternating series, etc. Along these lines, there is the following theorem:

Theorem 3.5 (Cauchy-Hadamard)**.** Suppose the coefficients of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ satisfy $\lim_{n\to\infty} |c_n|^{1/n} = L$, where L is a nonnegative real number or ∞. Then the radius of convergence of the power series is $R = \frac{1}{L}$ (by convention, if $L=0$ then $R=\infty$, and if $L=\infty$ then $R=0$).

Moreover, we can attempt to describe the radius of convergence of a sum of two power series. Suppose we have two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ with radii of convergence R_c and R_d, respectively (possibly ∞). Then the power series $\sum_{n=0}^{\infty}(c_n + d_n)x^n$ has radius of convergence R_{cd} satisfying $R_{cd} \geq \min(R_c, R_d)$, with equality holding if R_c and R_d are distinct. On the other hand, if $R_a = R_b$, then it is possible for R_{cd} to be strictly greater than $\min(R_c, R_d)$. (consider $\sum_{n=0}^{\infty} -x^n$ and $\sum_{n=0}^{\infty}(1 + (1/2)^n)x^n$; both of these series have radius of convergence 1, but their sum $\sum_{n=0}^{\infty}(1/2)^n x^n$ has radius of convergence $2 > \min(1, 1)$).

The importance of the interval of convergence is as follows: we can consider a power series as a *function* of x within the interval of convergence. Here is a basic example: consider the power series $\sum_{n=0}^{\infty} x^n$. This is a power series with interval of convergence $(-1, 1)$, but also a geometric series with sum $\frac{1}{1-x}$ on that interval. We may say that the power series gives a *(power series) representation* of $\frac{1}{1-x}$ on the interval of convergence.

Definition 3.6. If a function $f(x)$ equals $\sum c_n(x-a)^n$ on the interval of convergence of the power series, we say the function is a representation of (or just "is equal to") the power series centered, or expanded, at a.

This is extremely useful, because we can now switch between different representations of a function (its "formula" versus its series expansion). Moreover, the series representation of a function expanded at $x = a$ is *unique* (one has to give some more details about what this means, but let's forget about that for now).

We can also differentiate and integrate power series term-by-term: the derivative of $\sum c_n(x-a)^n$ is simply $\sum c_n \frac{d}{dx}(x-a)^n$, and similarly, the integral is $C + \sum c_n \int (x-a)^n dx$. Moreover, the radius of convergence stays the same (as the radius of convergence of the original power series) if we perform term-by-term differentiation or integration.

Example 3.7. The integral of $\frac{1}{1-x^{10}}$, which is hard to do analytically, becomes

$$\int \frac{1}{1-x^{10}} dx = \int \sum_{n=0} x^{10n} dx = C + \sum_{n=0} \frac{x^{10n+1}}{10n+1}.$$

The radius of convergence of the resulting power series is still 1, because that is the radius of convergence of the original power series.

Example 3.8. We can find a series representation of $\log(1+x)$ expanded at $x = 0$: since $\log(1+x)$ is an indefinite integral of $\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n x^n$, we have

$$\log(1+x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

To find what C is, we can just plug in the center point $x = 0$; the left-hand side is $\log(1) = 0$, while the right hand side is just C. So C must be 0, and thus $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$. You can check directly, or via the ratio test, that this series has radius of convergence 1.

Finally, this is a useful theorem that allows you to skip a lot of the work for finding radius of convergence of a power series, if you know that the power series represesents a function $f(x)$:

Theorem 3.9. In "nice" cases (such as anything you'll encounter in this class), the radius of convergence of a power series centered at a is equal to the distance from a to the nearest complex(!) singularity of the function f defined by the power series. By a singularity at c, we mean that $\lim_{x\to c} f(x)$ is ∞ or $-\infty$, so in particular, removable discontinuities don't count.

Example 3.10. In the previous example, we don't even need to know the power series of $\log(1+x)$ at 0 to know that it has radius of convergence 1. This is because $\log(1+x)$ has a singularity at -1 (as $\lim_{x\to -1^+} \log(1+x) = -\infty$), and that is the nearest singularity to 0, so the distance between 0 and the nearest singularity is 1.

Example 3.11. Consider the power series $1-2^2x^2+2^4x^4-2^6x^6+\ldots$, centered at 0. Via the ratio or root test, we can see that the radius of convergence is $R=\frac{1}{2}$. On the other hand, this power series is a geometric series that represents the function $f(x)=\frac{1}{1+(2x)^2}$ whenenver it converges. The singularities of f occur when its denominator $1+(2x)^2$ is 0, which happens when $x=\frac{i}{2}$ of $x=-\frac{i}{2}$. The distance from 0 to either $\frac{i}{2}$ or $-\frac{i}{2}$ is $\frac{1}{2}$, the same as the radius of convergence of the power series.

Notice that even though $f(x)$ has no *real* singularities, the radius of convergence is not ∞. This shows that we must also consider complex singularities when applying this heuristic.

Example 3.12. Consider the power series $\sum_{n=0}^{\infty} c_n x^n$ with coefficients defined recursively: $c_0=0$, $c_1=1$, and $c_n=c_{n-1}+c_{n-2}$ for $n\geq 2$. Unless you already know the limit $\lim_{n\to\infty}\frac{c_{n+1}}{c_n}$, it would be impossible to find the radius of convergence using methods involving the ratio test. So we need an alternative approach.

Suppose the series represents a well-defined function $f(x)$. Then

$$x^2 f(x) = \sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=2}^{\infty} c_{n-2} x^n,$$

and

$$xf(x) = \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{n=1}^{\infty} c_{n-1} x^n = c_0 x + \sum_{n=2}^{\infty} c_{n-1} x^n.$$

Hence

$$x^2 f(x) + xf(x) = c_0 x + \sum_{n=2}^{\infty} (c_{n-1}+c_{n-2}) x^n = c_0 x + \sum_{n=2}^{\infty} c_n x^n = c_0 x + (f(x) - c_1 x - c_0).$$

Since we know that $c_0 = 0$ and $c_1 = 1$, we conclude that $x^2 f(x) + fx(x) = f(x) - x$, or upon rearranging, $f(x) = \frac{-x}{x^2+x-1}$. The singularities of f occur when its denominator $x^2 + x - 1$ is 0, which happens when $x = \frac{1+\sqrt{5}}{2}$ of $x = \frac{1-\sqrt{5}}{2}$ after solving the quadratic. The singularity at $\frac{1-\sqrt{5}}{2}$ is closer to the center 0 (because $\left|\frac{1-\sqrt{5}}{2}\right| = \frac{-1+\sqrt{5}}{2} < \frac{1+\sqrt{5}}{2} = \left|\frac{1+\sqrt{5}}{2}\right|$), so this is the minimum distance from the center to a singularity of f. Therefore $R = \left|\frac{1-\sqrt{5}}{2}\right| = \frac{\sqrt{5}-1}{2}$.

4 Taylor Series

The most important example of power series are Taylor series, defined as follows:

Definition 4.1. Suppose f is an infinitely differentiable function (e.g. all the functions you've seen in this class). The *Taylor series* of f at a is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor coefficient is therefore $c_n = \frac{f^{(n)}(a)}{n!}$. The *$N$th Taylor polynomial* is the Nth partial sum $T_N(x) := \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$. A *Maclaurin series* is a Taylor series centered at 0.

Intuitively, the Nth Taylor polynomial is the "best Nth-degree polynomial approximation to $f(x)$ at a". You can check for yourself that when $N = 1$, you recover the tangent line of f at a. However, note that even though we say that the Taylor polynomials approximate f, there is no *a priori* reason why the series $\frac{f^{(n)}(a)}{n!}(x-a)^n$ should converge to f (locally near a given x); in fact this is not always the case. If the series does converge to the true function f on an open interval about any fixed x, we call f *analytic*.

To find a Taylor series expansion of $f(x)$ at a, you just need to calculate all the derivatives of f at a, and plug them into the definition. Alternatively, if you have a series representation for f at a via other means (e.g. the series for $\log(1+x)$ found in Example 3.8), by the *uniqueness* of power series near a point, it follows that such a series must be the Taylor series as well!

Remark 4.2. You should know the Taylor series at $x = 0$ for the common functions e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$, $\log(1+x)$, and $\arctan(x)$. You should also know the binomial series expansion at $x = 0$:

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots,$$

valid for any real number k.

Using uniqueness of power series expansions, we can also construct new Taylor series from old ones. Here are some typical examples.

Example 4.3. Since the Taylor series of e^x at $x = 0$ is $1 + x + \frac{x^2}{2!} + \dots$, the Taylor series of e^{-x^2} at $x = 0$ is $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$.

Example 4.4. Since the Taylor series of $\log(1 + x)$ at $x = 0$ is $x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$, we can compute the Taylor series of $(x - 5)\log(x - 4)$ at $x = 5$. Indeed, we have

$$(x-5)\log(1+(x-5)) = (x-5)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-5)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-5)^{n+1}}{n} = \sum_{n=2}^{\infty}\frac{(-1)^n(x-5)^n}{n-1}.$$

Example 4.5. Consider $\cos(x)\arctan(x)$. We could find the first few terms of the Taylor series at $x = 0$ by repeated differentiation, but that would be tedious and probably lead to errors. However, since we know the Taylor series for cosine and arctan, we can multiply them together:

$$\begin{aligned}\cos(x)\arctan(x) &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)\\ &= x + x^3\left(-\frac{1}{3} - \frac{1}{2!}\right) + x^5\left(\frac{1}{5} + \frac{1}{2!\cdot 3} + \frac{1}{4!}\right) + \dots\\ &= x - \frac{5x^3}{6} + \frac{49x^5}{120} + \dots.\end{aligned}$$

Indeed, the only way to get a linear term is by multiplying 1 by x. We can't get any quadratic terms, and the only ways to get cubic terms are from $1\cdot(-x^3/3)$ and $(-x^2/2!)\cdot(x)$. We can't get any x^4 terms, and the only ways to get x^5 terms are from $1\cdot(x^5/5)$, $(-x^2/2!)\cdot(-x^3/3)$, and $(x^4/4!)\cdot(x)$. This analysis can be continued indefinitely.

Example 4.6. Let's compute the limit $\lim_{x\to 0}\frac{\cos(x^3)\arctan(x^3)-x^3}{x^9}$. We can deduce as in the above example that

$$\cos(x^3)\arctan(x^3) = x^3 - \frac{5x^9}{6} + \frac{49x^{15}}{120} + \dots.$$

Hence

$$\frac{\cos(x^3)\arctan(x^3) - x^3}{x^9} = -\frac{5}{6} + \frac{49x^6}{120} + \dots,$$

where the ellipses represent higher-order terms. So the limit as $x \to 0$ is $-\frac{5}{6}$, as all higher-order terms vanish (they involve positive powers of x, which goes to 0).

4.1 Taylor Series Error Bounds

Since Taylor series are used as approximations, it is important to be able to give error bounds for the approximation given by the Nth Taylor polynomial. The most precise theorem is as follows:

Theorem 4.7 (Taylor Remainder Theorem)**.** Suppose f is an infinitely differentiable function (e.g. all the functions you've seen in this class), and consider its Taylor series centered at a. Then for any x and any $N \geq 0$, we may write

$$f(x) = T_N(x) + \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c in the open interval between x and a (so (x,a) if $x < a$, and (a,x) if $x > a$).

Remark 4.8. You can check for yourself that when $N = 0$, the above statement reduces to the mean value theorem.

Corollary 4.9. The version of the Taylor Remainder Theorem that you might have seen is as follows: use the same notation as in Theorem 4.7, and let $E_N(x)$ be the absolute error between the original function and the Nth Taylor polynomial evaluated at x, so $E_N(x) = |f(x) - T_N(x)|$. Then if the magnitude of the $(N+1)$-st derivative of f, $|f^{(N+1)}(x)|$ is bounded by some constant M for all x such that $|x-a| \leq d$, then for such x, $E_N(x) \leq \frac{M}{(N+1)!}|x-a|^{N+1}$.

This statement follows from Theorem 4.7 (make sure you see why).

Here are some typical applications of this error-bounding theorem.

Example 4.10. We will find how large N has to be such that the Nth Taylor approximation of $e^{0.2}$ is within 10^{-10} of the true value. Using the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ expanded at 0, and the fact that $\frac{d^{N+1}}{dx^{N+1}}(e^x) = e^x$ is bounded by $M := e^{0.2}$ on $(0, 0.2)$, we simply need to find the smallest possible positive integer N such that

$$\frac{e^{0.2}}{(N+1)!}|0.2 - 0|^{N+1} = \frac{e^{0.2}}{(N+1)!} \cdot 0.2^{N+1}$$

is less than 10^{-10}, since we would then have the chain of inequalities

$$|E_N(0.2)| \leq \frac{e^{0.2}}{(N+1)!} \cdot 0.2^{N+1} \leq 10^{-10}.$$

Using a calculator, we find that $N = 7$ is the smallest such N.

Example 4.11. Using similar techniques, we can prove that $f(x) = e^x$ is analytic at any real number a. To do this, we need to prove that on the interval $I_a := (a-1, a+1)$ about any real a, the Taylor series $e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$ at a converges to f on that interval. This is equivalent to showing that the Nth absolute error $E_N(x)$ converges to 0 (as $N \to \infty$) for any x in I_a.

Note that on I_a and any $N \geq 0$, we have $\frac{d^{N+1}}{dx^{N+1}} f(x) = e^x$, so that the $(N+1)$-st derivative is bounded by e^{a+1} on the interval. Therefore for x in I_a, the Nth absolute error $E_N(x)$ satisfies

$$E_N(x) \leq \frac{e^{a+1}}{(N+1)!}|x-a|^{N+1} \leq \frac{e^{a+1}}{(N+1)!},$$

because $|x-a| \leq 1$. But as $N \to \infty$, $\frac{e^{a+1}}{(N+1)!}$ converges to 0, so by the squeeze rule, $E_N(x) \xrightarrow{N\to\infty} 0$ as well, which is what we wanted to show.